International Journal of Current Research and Techniques (IJCRT)
E-ISSN: 2349-3194 (Online) | P-ISSN: 2348-4446 (Print)
UGC Approved Journal No: 47722 [No: 2616]
DOI: https://doi.org/10.5281/ZENODO.17119483

# Generalized Hilbert-Type operator in Weighted Bergman Spaces

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*Manuscript Received: Aug 19, 2025; Revised: Sep 03, 2025; Published: Sep 15, 2025*

**Abstract:** If $f$ be an analytic function on the unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$. We consider the generalized Hilbert-type operators defined by

$$\mathcal{H}_{a,b}(f)(z) = \frac{\Gamma(a+1)}{\Gamma(b+1)} \int_0^1 \frac{f(t)(1-t)^b}{(1-tz)^{a+1}} dt,$$

where $a$ and $b$ are non-negative real numbers. In particular, for $a = b = \beta$, $\mathcal{H}_{a,b}$ becomes the Generalized Hilbert operator $\mathcal{H}_\beta$ and for $\beta = 0$, it becomes the classical Hilbert operator $\mathcal{H}$. We study these operators acting on weighted Bergman spaces $A_p^\alpha$ of analytic functions in $\mathbb{D}$. Here we give conditions on the parameters $a$, $b$, $p$ and $\alpha$ such that $\mathcal{H}_{a,b}$ is bounded on $A_p^\alpha$.

**Keywords:** Generalized Hilbert-type operators, Integral-type operators, Bergman Spaces

**1. Introduction:**

Let $\mathbb{D}$ denote the unit disk in the complex plane and let $H(\mathbb{D})$ be the class of all analytic functions on $\mathbb{D}$. Let $f \in H(\mathbb{D})$ and $r < 1$. For $0 < p < \infty$, the integral means of $f$ are defined by

$$M_p(r,f) = \left(\frac{1}{2\pi}\int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^p d\theta\right)^{\frac{1}{p}}$$

and

$$M_\infty(r,f) = \max_{0\le\theta\le 2\pi} \left|f\left(re^{i\theta}\right)\right|.$$

$M_p\,(r,f)$ is an increasing function of $r$.

We recall that the Hardy space $H^p, 1 \le p \le \infty$, of the unit disk $\mathbb{D}$ is the Banach space of analytic functions $f: \mathbb{D} \to \mathbb{C}$ such that

$$\|f\|_{H^p} = \sup_{r<1}\left(\int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^p \frac{d\theta}{2\pi}\right)^{\frac{1}{p}} < \infty,$$

for finite $p$, and

$$\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|.$$

More details on Hardy spaces can be found in [1]. For $0 < p < \infty$, the Bergman space $A^p$ consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{A^p}^p = \int_{\mathbb{D}} |f(z)|^p dm(z) < \infty,$$

where $dm(z) = \frac{1}{\pi} r dr d\theta$ is the normalized Lebesgue area measure on $\mathbb{D}$. We refer to [8] for more details on Bergman spaces. For $0 < p < \infty$ and $\alpha > -1$, the weighted Bergman space $A_p^\alpha$ consists of all $f \in H(\mathbb{D})$, such that

$$\|f\|_{A_p^\alpha}^p = (\alpha+1)\int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\alpha dm(z) < \infty.$$

Equivalently,

$$\|f\|_{A_p^\alpha}^p = (\alpha+1)\int_0^1 M_p^p(r,f)(1-r)^\alpha dr < \infty.$$

For $\alpha = 0$, we get the Classical Bergman space. For $p \in \mathbb{R}$, the Dirichlet-type space $S^p$ consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{S^p}^2 = \sum_{n=0}^{\infty}(n+1)^p|a_n|^2 < \infty$$

where $f$ has the Taylor expansion $f(z) = \sum_{n=1}^{\infty} a_n z^n$. Recall that $S^p$ is a Hilbert space with inner product

$$\langle f,g\rangle = \sum_{n=0}^{\infty}(n+1)^p a_n\overline{b_n}$$

where $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$. For $p = 0$, we get the Hardy space $H^2$, for $p = -1$ the Bergman space $A^2$ and for $p = 1$ the Dirichlet Space $\mathcal{D}$. Refer [6] for more details. If $0 < p < 2$ and $f \in S^p$, then

$$c_p\|f\|_{S_p}^2 \le |f(0)|^2 + 2\int_0^1 r(1-r^2)^{1-p}M_2^2(r,f')dr \le C_p\|f\|_{S_p}^2.$$

In [7] , authors gave the optimal values of $c_p$ and $C_p$.

The Hilbert matrix $H$ with entries $a_{i,j} = \frac{1}{i+j+1}$, (where $i,j$ are positive integers) induces an operator by multiplication on sequences

$$H: (a_n)_{n\ge 0} \to \left(\sum_{k=0}^{\infty}\frac{a_k}{n+k+1}\right)_{n\ge 0}.$$

For $1 < p < \infty$ Hilbert's inequality

$$\left(\sum_{n=0}^{\infty}\left|\sum_{k=0}^{\infty}\frac{a_k}{n+k+1}\right|^p\right)^{\frac{1}{p}} \le \frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\left(\sum_{k=0}^{\infty}|a_k|^p\right)^{\frac{1}{p}}$$

guarantees that $H$ induces a bounded operator on $l^p$ spaces with norm

$$\|H\|_{l^p\to l^p} = \frac{\pi}{\sin\left(\frac{\pi}{p}\right)}.$$

The constant $\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}$ is the best-possible [4]. The Hilbert matrix $H$ induces an operator $\mathcal{H}$ on spaces of analytic functions. For $f \in H(\mathbb{D})$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$, Hilbert operator $\mathcal{H}(f)$ is defined by

$$\mathcal{H}(f)(z) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\frac{a_k}{n+k+1}\right)z^n \qquad (1.1)$$

If $f \in H^1$, then, by Hardy's inequality [1] we have

$$\sum_{n=0}^{\infty}\frac{|a_n|}{n+1} \le \pi\|f\|_{H^1},$$

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for which it follows that the power series (1.1) has bounded coefficients and consequently $\mathcal{H}(f)(z)$ is well defined on the unit disk [6]. The Hilbert operator has an integral representation given by

$$\mathcal{H}(f)(z) = \int_0^1 \frac{f(t)}{1-tz}dt.$$

The Hilbert operator had been studied on Hardy spaces by E. Diamantopolous and A.G. Siskakis in [4] and in [3] by M. Dostanić, M. Jevtić and D. Vukotić. In [4] authors proved that $\mathcal{H}(f)(z)$ is a bounded on the Hardy spaces $H^p$ for$p > 1$ but not on $H^1$ and $H^\infty$. They proved the following result:

**Theorem 1**.

1. *If* $2 \leq p < \infty$ *then* $\|\mathcal{H}(f)\|_{H^p} \leq \frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\|f\|_{H^p}$ *for each* $f \in H^p$.
2. *If* $1 < p < 2$, *then* $\|\mathcal{H}(f)\|_{H^p} \leq \frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\|f\|_{H^p}$ *for each* $f \in H^p$ *with* $f(0) = 0$.

Boundedness of $\mathcal{H}(f)(z)$ on Bergman spaces $A^p$ for $2 < p < \infty$ was proved in [2] by E. Diamantopolous. He showed that $\mathcal{H}(f)(z)$ is not even defined in $A^2$. More precisely the author proved the following theorem:

**Theorem 2**. *The operator* $\mathcal{H}$ *is bounded on Bergman spaces* $A^p, 2 < p < \infty$, *and satisfies:*

1. *If* $4 \leq p < \infty$ *and* $f \in A^p$, *then* $\|\mathcal{H}(f)\|_{A^p} \leq \frac{\pi}{\sin\left(\frac{2\pi}{p}\right)}\|f\|_{A^p}$.
2. *If* $2 < p < 4$ *and* $f \in A^p$, *then* $\|\mathcal{H}(f)\|_{A^p} \leq \left(\frac{2^{7-p}}{9(p-2)} + 2^{4-p}\right)\frac{\pi}{\sin\left(\frac{2\pi}{p}\right)}\|f\|_{A^p}$.
3. *If* $2 < p < 4$ *and* $f \in A^p$, *with* $f(0) = 0$, *then* $\|\mathcal{H}(f)\|_{A^p} \leq \left(\frac{p}{2}+1\right)^{\frac{1}{p}}\frac{\pi}{\sin\left(\frac{2\pi}{p}\right)}\|f\|_{A^p}$.

In 2009, S. Li and S. Stević for $\beta \geq 0$ defined Generalized Hilbert operator as follows:

$$\mathcal{H}_\beta(f)(z) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\frac{\Gamma(n+\beta+1)\Gamma(n+k+1)}{\Gamma(n+1)\Gamma(n+k+\beta+2)}a_k\right)z^n.$$

For $\beta = 0$, $\mathcal{H}_\beta(f)(z)$ gives Hilbert operator. $\mathcal{H}_\beta(f)(z)$ has an integral representation:

$$\mathcal{H}_\beta(f)(z) = \int_0^1 \frac{f(t)(1-t)^\beta}{(1-tz)^{\beta+1}}dt.$$

In [6] the authors proved the boundedness of generalized Hilbert operator on Hardy spaces in polydisk. In [5] S. Li proved the boundedness of the generalized Hilbert operator on Dirichlet-type spaces $S^p$ for $\beta \geq 0$ and $0 < p < 1$. In 2015, S. Naik And K. Rajbangshi extended the class of generalized Hilbert operator by defining the operator

$$\mathcal{H}_{a,b}(f)(z) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\frac{\Gamma(n+a+1)\Gamma(n+k+1)}{\Gamma(n+1)\Gamma(n+k+b+2)}a_k\right)z^n.$$

and called it the Generalized Hilbert-Type operator [7]. For $a = b = \beta, \mathcal{H}_{a,b} = \mathcal{H}_\beta$ and $\mathcal{H}_{a,b} = \mathcal{H}$ for $a = b = 0$. $\mathcal{H}_{a,b}$ has an integral representation as follows:

$$\mathcal{H}_{a,b}(f)(z) = \frac{\Gamma(a+1)}{\Gamma(b+1)}\int_0^1 \frac{f(t)(1-t)^b}{(1-tz)^{a+1}}dt.$$

They proved the boundedness of $\mathcal{H}_{a,b}$ on the Dirichlet-type spaces $S^p$ and the Bergman spaces. They extend and improve the result of boundedness of $\mathcal{H}_\beta$ on $S^p$ spaces by proving the boundedness of $\mathcal{H}_{a,b}$ in $S^p(0 < p < 2)$and finding the norm of the operator $\mathcal{H}_{a,b}$ on $S^p$. They also gave conditions on $a, b$ and $p$ such that $\mathcal{H}_{a,b}$ is bounded on $A^p$ for $p > 2$.

The main objective of this article is to prove the boundedness of Generalized Hilbert-type operator on the weighted Bergman spaces $A_p^\alpha$ for certain values of the parameters. We use the representation of $\mathcal{H}_{a,b}$ in terms of the weighted composition operator to prove the main results in Lemma 2.2 and Theorem 2.3. Lemma 2.1 is established to prove Lemma 2.2. The results will extend the results on the boundedness of Hilbert-type operators on the Bergman spaces.

Throughout this article, $C$ means a positive constant, which may differ from one occurrence to the other.

## 2. Generalized Hilbert-type operator on $\mathrm{A}_\mathrm{p}^\alpha$ spaces

In this section, we present the main results of the article. For that purpose, the representation $\mathcal{H}_{a,b}$ in terms of a weighted composition operator is used as follows [7]: For $z \in \mathbb{D}$, we choose the contour

$$\zeta(t) = \zeta_z(t) = \frac{t}{(t-1)z+1}, 0 < t < 1.$$

A change in variable in the integral representation of $\mathcal{H}_{a,b}$ gives

$$\mathcal{H}_{a,b}(f)(z) = \frac{\Gamma(a+1)}{\Gamma(b+1)} \int_0^1 f\left(\frac{t}{(t-1)z+1}\right) \times \frac{(1-t)^b(1-z)^{b-a}}{(1+(t-1)z)^{b-a+1}} dt$$

In [7] the authors defined a weighted composition operator as follows:

$$T_t(f)(z) = f(\phi_t(z))\omega_t^{b-a+1}(z),$$

where $\phi_t(z) = \frac{t}{(t-1)z+1}$ and $\omega_t(z) = \frac{1}{(t-1)z+1}$. Then

$$\mathcal{H}_{a,b}(f)(z) = \frac{\Gamma(a+1)}{\Gamma(b+1)} \int_0^1 T_t(f)(z)(1-t)^b(1-z)^{b-a} dt$$

**Lemma 3**. *For* $0 < p < \infty, \alpha > -1$ *and* $f \in A_p^\alpha$ $|f(z)| \le C \frac{\|f\|_{A_p^\alpha}}{(1-|z|)^{\frac{\alpha+2}{p}}}, z \in \mathbb{D}$.

*Proof.* Using the definition of the weighted Bergman space and increasing property of integral means we have

$$\begin{aligned}
\|f\|_{A_p^\alpha}^p &= \int_0^1 M_p^p(r,f)(1-r)^\alpha dr \\
&\ge \int_{\frac{1+|z|}{2}}^{\frac{3+|z|}{4}} (1-r)^\alpha M_p^p(r,f) dr \\
&\ge M_p^p\left(\frac{1+|z|}{2}, f\right)\left(\frac{1-|z|}{4}\right)^\alpha \int_{\frac{1+|z|}{2}}^{\frac{3+|z|}{4}} dr \\
&= C M_p^p\left(\frac{1+|z|}{2}, f\right)(1-|z|)^{\alpha+1}.
\end{aligned}$$

Applying Cauchy's integral formula to any $p \in (0, \infty)$, we have

$$(1-|z|)|f(z)|^p \le (1+|z|)^\alpha M_p^p\left(\frac{1+|z|}{2}, f\right).$$

From (2.3) and (2.4), we get

$$\|f\|_{A_p^\alpha}^p \ge C(1-|z|)|f(z)|^p(1-|z|)^{\alpha+1} = C|f(z)|^p(1-|z|)^{\alpha+2}$$

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which gives us

$$|f(z)|^p \leq C \frac{\|f\|^p_{A_p^\alpha}}{(1-|z|)^{\alpha+2}}.$$

**Lemma 4**. *Suppose* $f \in A_p^\alpha, \alpha > -1$. *If* $\alpha + 2 > p(b-a)$, *then* $\|T_t f\|_{A_p^\alpha} \leq C(1-t)^{\frac{-\alpha-2}{p}} \|f\|_{A_p^\alpha}$.

*Proof.* From (2.1), using the definition of the weighted Bergman spaces we have

$$\|T_t f\|^p_{A_p^\alpha} = \int_{\mathbb{D}} \left|f(\phi_t(z))\omega_t^{b-a+1}(z)\right|^p (1-|z|^2)^\alpha dm(z)$$

Using Lemma 2.1 we have

$$\begin{aligned}
\|T_t f\|^p_{A_p^\alpha} &\leq C \quad \|f\|^p_{A_p^\alpha} \int_{\mathbb{D}} \left|\omega_t^{b-a+1}(z)\right|^p \frac{(1-|z|^2)^\alpha}{(1-|\phi_t(z)|)^{\alpha+2}} dm(z) \\
&\leq C \quad \|f\|^p_{A_p^\alpha} \int_{\mathbb{D}} \frac{dm(z)}{|1+(t-1)z|^{(b-a+1)p-\alpha-2}(1-t)^{\alpha+2}(1-|z|)^2} \\
&= C \quad \|f\|^p_{A_p^\alpha}(1-t)^{p-\alpha-2} \int_{\mathbb{D}} \frac{|1+(t-1)z|^{\alpha+2-(b-a+1)p}}{(1-|z|)^2} dm(z) \\
&\leq C \quad \|f\|^p_{A_p^\alpha}(1-t)^{-\alpha-2} \int_{\mathbb{D}} \frac{|1+(t-1)z|^{\alpha+2-(b-a)p}}{(1-|z|\,)^{2+p}} dm(z).
\end{aligned}$$

If $\alpha + 2 > p(b-a)$, then

$$\begin{aligned}
\|T_t f\|^p_{A_p^\alpha} &\leq C \quad \|f\|^p_{A_p^\alpha}(1-t)^{-\alpha-2} \int_{\mathbb{D}} \frac{dm(z)}{(1-|z|\,)^{2+p}} \\
&\leq C \quad \|f\|^p_{A_p^\alpha}(1-t)^{-\alpha-2} \int_0^{2\pi}\int_0^1 \frac{\frac{1}{\pi} r dr d\theta}{(1-r)^{2+p}} \\
&\leq C \quad \|f\|^p_{A_p^\alpha}(1-t)^{-\alpha-2}.
\end{aligned}$$

**Theorem 5**. *Suppose* $f \in A_p^\alpha, \alpha > -1$. *If* $p(b+1) > \alpha + 2 > p(b-a)$ *and* $b \geq a$, *then* $\mathcal{H}_{a,b}$ *is bounded on* $A_p^\alpha$.

*Proof.* From (2.2) we have

$$\begin{aligned}
\|\mathcal{H}_{a,b}\|^p_{A_p^\alpha} &= \int_{\mathbb{D}} \left|\mathcal{H}_{a,b} f\right|^p (1-|z|^2)^\alpha dm(z) \\
&= \int_{\mathbb{D}} \left|\int_0^1 (1-t)^b (1-z)^{b-a} T_t f(z) dt\right|^p (1-|z|^2)^\alpha dm(z) \\
&= \left[\left\{\int_{\mathbb{D}} \left|\int_0^1 (1-t)^b (1-z)^{b-a} T_t f(z)(1-|z|^2)^{\frac{\alpha}{p}} dt\right|^p dm(z)\right\}^{\frac{1}{p}}\right]^p \\
&\leq \left[\int_0^1 \left\{\int_{\mathbb{D}} \left|(1-t)^b (1-z)^{b-a} T_t f(z)(1-|z|^2)^{\alpha/p}\right|^p dm(z)\right\}^{1/p} dt\right]^p \\
&= \left[\int_0^1 \left\{\int_{\mathbb{D}} (1-t)^{bp} |1-z|^{p(b-a)} |T_t f(z)|^p (1-|z|^2)^\alpha dm(z)\right\}^{1/p} dt\right]^p.
\end{aligned}$$

If $b \geq a$ then

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$$\|\mathcal{H}_{a,b}\|_{A_p^\alpha}^p \leq C\left[\int_0^1 \left\{\int_{\mathbb{D}} |T_t f(z)|^p \,(1-|z|^2)^\alpha dm(z)\right\}^{\frac{1}{p}} (1-t)^b dt\right]^p$$

$$= C \left[\int_0^1 \|T_t f\|_{A_p^\alpha}(1-t)^b dt\right]^p$$

Using Lemma 2.2 we have

$$\|\mathcal{H}_{a,b}\|_{A_p^\alpha}^p \leq C\left[\int_0^1 (1-t)^{b-\frac{\alpha+2}{p}} dt\right]^p \|f\|_{A_p^\alpha}^p$$

For different values of the parameters, we have the following corollary:

**Corollary 6**. *Suppose* $f \in A_p^\alpha, \alpha > -1$. *If* $p(b+1) > \alpha + 2$, *then* $\mathcal{H}_\beta$ *is bounded on* $A_p^\alpha$. *In particular,* $\mathcal{H}_\beta$ *and* $\mathcal{H}$ *are bounded on Bergman spaces.*

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